

Time Dependent Boundary Conditions for Hyperbolic Systems

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Time dependent numerical models for hyperbolic systems, such as the fluid dynamics equations, require time dependent boundary conditions when the systems are solved in a finite domain. The "correct" boundary condition depends on the external solution, but for many problems the external solution is not known. In such cases nonreflecting boundary conditions often produce solutions with the desired behavior. This paper extends the concept of non-reflecting boundary conditions to the multidimensional case in non-rectangular coordinate systems. Results are given for several fluid dynamics test problems: the traveling shock wave, shock tube, spherical explosion, and homologous expansion problems in one dimension, and a traveling shock wave moving at a 45° angle with respect to the x axis in two dimensions.

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1. INTRODUCTION

Numerical solutions to hyperbolic systems of differential equations, such as the fluid dynamics equations, are usually obtained over a finite region. The time evolution of the system is governed not only by the state in the interior of the region, but also by waves which enter the region from outside its boundary. Thus boundary conditions which describe the incoming waves are required to completely specify the behavior of the system. The outgoing waves are described by characteristic equations, while the incoming waves may often be specified by a non-reflecting boundary condition. Nonreflecting boundary conditions for multidimensional problems are described below.

2. WAVES IN ONE DIMENSION

Consider first the one dimensional case in orthogonal (but not necessarily rectangular) coordinates. We have a system of n equations describing the behavior of n dependent variables. Let $\tilde{\mathbf{U}}$ be the vector of conservative variables, satisfying

$$\frac{\partial \tilde{\mathbf{U}}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \mathbf{C}' = 0, \quad (1)$$

where \mathbf{F} is the flux vector, and \mathbf{C}' is an inhomogeneous term not containing derivatives (which often arises from divergence terms in nonrectangular geometries). Equation (1) describes the conservation properties of the system; that is, it relates the rate of change of the integral of a field over a small volume to the flux of that field across the volume boundaries.

An alternate form for Eq. (1) is the primitive system, with a vector of dependent variables \mathbf{U} , which satisfies

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x} + \mathbf{C} = 0, \quad (2)$$

where \mathbf{A} is an $n \times n$ matrix. The choice of primitive variable vector \mathbf{U} is not unique (although the choice of conserved variables is), and could be defined as the conservative vector. The following analysis assumes that \mathbf{U} and $\tilde{\mathbf{U}}$ are distinct.

The two systems are related by

$$\frac{\partial \tilde{\mathbf{U}}}{\partial t} = \mathbf{P} \frac{\partial \mathbf{U}}{\partial t}, \quad (3)$$

$$\frac{\partial \mathbf{F}}{\partial x} = \mathbf{Q} \frac{\partial \mathbf{U}}{\partial x}, \quad (4)$$

with

$$P_{ij} = \frac{\partial \tilde{U}_i}{\partial U_j}, \quad (5)$$

$$Q_{ij} = \frac{\partial F_i}{\partial U_j}, \quad (6)$$

and

$$\mathbf{A} = \mathbf{P}^{-1} \mathbf{Q}, \quad (7)$$

$$\mathbf{C} = \mathbf{P}^{-1} \mathbf{C}', \quad (8)$$

where \mathbf{P} and \mathbf{Q} are also $n \times n$ matrices.

Now let \mathbf{l}_i and \mathbf{r}_i be the set of left (row) and right (column) eigenvectors of \mathbf{A} , satisfying

$$\mathbf{l}_i \mathbf{A} = \lambda_i \mathbf{l}_i, \quad (9)$$

$$\mathbf{A} \mathbf{r}_i = \lambda_i \mathbf{r}_i, \quad (10)$$

where the λ_i are the n eigenvalues of \mathbf{A} , ordered so that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. (The system is hyperbolic if the eigenvalues of \mathbf{A} are real.) Then we obtain a diagonal matrix Λ by the similarly transformation

$$\mathbf{S} \mathbf{A} \mathbf{S}^{-1} = \Lambda, \quad (11)$$

where the rows of \mathbf{S} are the left eigenvectors \mathbf{l}_i , the columns of \mathbf{S}^{-1} are the right eigenvectors \mathbf{r}_i , and the matrix \mathbf{A} is diagonal, with $\mathbf{A}_{ii} = \lambda_i$. (Note that the transformation follows from the orthogonality of the normalized left and right eigenvectors: $\mathbf{l}_i \mathbf{r}_j = \delta_{ij}$.)

Multiplying Eq. (2) by \mathbf{S} gives

$$\mathbf{S} \frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \mathbf{S} \frac{\partial \mathbf{U}}{\partial x} + \mathbf{S} \mathbf{C} = 0, \quad (12)$$

or

$$\mathbf{l}_i \frac{\partial \mathbf{U}}{\partial t} + \lambda_i \mathbf{l}_i \frac{\partial \mathbf{U}}{\partial x} + \mathbf{l}_i \mathbf{C} = 0, \quad (13)$$

in component form. Equation (12) is the characteristic equation corresponding to the original forms (1) and (2).

If we can define a new function \mathbf{V} by

$$dV_i = \mathbf{l}_i d\mathbf{U} + \mathbf{l}_i \mathbf{C} dt, \quad (14)$$

then (13) becomes

$$\frac{\partial V_i}{\partial t} + \lambda_i \frac{\partial V_i}{\partial x} = 0, \quad (15)$$

which is a set of wave equations for waves with characteristic velocities λ_i . Each wave amplitude V_i is constant along the curve C_i in the xt plane defined by $dx/dt = \lambda_i$.

However, the definition of (14) generally can be made only if \mathbf{A} and \mathbf{C} are constant everywhere, or if no more than two differentials appear on the right side of (14). Otherwise the coefficients in (14) must satisfy Pfaff's condition for the integrability of differential forms for the functions V_i to exist [1], a condition not met for the fluid equations. Nevertheless, the characteristic form of (13) holds true independent of (14).

3. NONREFLECTING BOUNDARY CONDITIONS IN ONE DIMENSION

In the one dimensional case we wish to solve Eq. (1) over the region $a \leq x \leq b$. The problem is an initial boundary value problem, because both initial data in the region $a \leq x \leq b$ and time dependent boundary conditions at $x = a, b$ are needed for the problem to be well posed. Difficulty arises in the boundary condition specification because Eq. (1) generally contains eigenvalues of both signs at the boundaries, implying that waves are propagating into and out of the domain. It is therefore more fruitful to work with the characteristic form at the boundaries, since we can then consider each wave separately.

The outgoing waves (those with $\lambda_i \leq 0$ at $x = a$, and $\lambda_i \geq 0$ at $x = b$) depend only on information at and within the boundaries. Thus those equations in the form of (13) which represent outgoing waves can be solved as is, or in any equivalent form. Properly designed numerical approximations to (13) for outgoing waves, which depend on one-sided finite difference approximations involving only interior and boundary points, will therefore be stable.

The incoming waves (with $\lambda_i > 0$ at $x = a$, and $\lambda_i < 0$ at $x = b$) are another matter. They depend on data exterior to the boundary, and numerical approximations to (13) not involving exterior data will be unstable. Thus we need to know something about the exterior solution in order to specify useful boundary conditions. In some problems, particularly steady state aerodynamics problems, the far field solutions are known to a good approximation, and the appropriate values can be specified [2, 3].

For time dependent problems (and many steady state problems as well) it is often desirable to use so-called nonreflecting or radiation boundary conditions, which have the property of minimizing reflections from outgoing waves. Bayliss and Turkel [4] formulated a perturbation approach in which the perturbations about the desired steady state were expressed in terms of waves. Then they imposed boundary conditions which annihilated the outgoing waves (i.e., prevented the generation of incoming waves). Engquist and Majda [5, 6] developed nonlocal, nonreflecting boundary conditions for linear systems. From their nonlocal conditions they derived a sequence of partially absorbing local conditions. Hedstrom [7] developed a nonreflecting boundary condition for the one dimensional rectangular, nonlinear case. As the only nonlinear condition, Hedstrom's is by far the most useful for time dependent problems. It will be generalized to multidimensional problems and non-rectangular coordinate systems below.

Hedstrom's nonreflecting boundary condition [7] can be stated in the following way: *the amplitudes of the incoming waves are constant, in time, at the boundaries.* This is the same as saying that there are no incoming waves, as it is the change in amplitude which indicates a wave. Mathematically, this condition is

$$\left. \frac{\partial V_i}{\partial t} \right|_{x=a,b} = 0, \quad (16)$$

in terms of the wave amplitude V_i of (14), or

$$\left(\mathbf{I}_i \frac{\partial \mathbf{U}}{\partial t} + \mathbf{I}_i \mathbf{C} \right) \Big|_{x=a,b} = 0, \quad (17)$$

in general, for those waves whose characteristic velocities are directed inward at the boundary. Equations (17) for the incoming waves and (13) for the outgoing waves completely determine the solution at the boundaries.

Note that Eq. (17) will not give the desired behavior for any problem which should in fact contain incoming waves. In such a case one must be able to specify

something about the incoming waves. Fortunately, Eq. (17) seems to be adequate for many problems of interest.

It is easy to write a general equation which automatically reduces to Eq. (17) or Eq. (13) for incoming and outgoing waves. The general form is

$$\left(\mathbf{I}_i \frac{\partial \mathbf{U}}{\partial t} + \mathcal{L}_i + \mathbf{I}_i \mathbf{C} \right) \Big|_{x=a,b} = 0, \quad (18)$$

where

$$\mathcal{L}_i = \begin{cases} \lambda_i \mathbf{I}_i \frac{\partial \mathbf{U}}{\partial x} & \text{for outgoing waves,} \\ 0 & \text{for incoming waves.} \end{cases} \quad (19)$$

Thus the characteristic and nonreflecting boundary conditions can be combined in a very natural way, unlike other extrapolation methods.

The set of Equations (18) is solved by a method of lines approach, in a way similar to (and along with) that for the conservative equations in the interior (as described in Section 6). The function values are obtained at the discrete coordinate positions x_i , where

$$x_i = a + i \Delta x, \quad (20)$$

$$\Delta x = (b - a)/I. \quad (21)$$

Equation (1) is solved at the interior points, defined by $0 \leq i \leq I$, while the boundary equation (in the form of Eq. (25) below) is solved at the boundary points, defined by $i < 0$ or $i > I$. (The interior scheme of Section 6 requires two boundary points at each boundary.) The spatial derivatives in (19) are evaluated using one-sided difference approximations

$$\left. \frac{\partial \mathbf{U}}{\partial x} \right|_i = \frac{1}{\Delta x} (\mathbf{U}_{i+1} - \mathbf{U}_i), \quad i < 0, \quad (22)$$

$$= \frac{1}{\Delta x} (\mathbf{U}_i - \mathbf{U}_{i-1}), \quad i > I. \quad (23)$$

To get an equation for the conservative variables, we first define \mathcal{L} as the column vector whose components are \mathcal{L}_i , and write

$$\mathbf{S} \frac{\partial \mathbf{U}}{\partial t} + \mathcal{L} + \mathbf{S} \mathbf{C} = 0, \quad (24)$$

which leads to

$$\frac{\partial \tilde{\mathbf{U}}}{\partial t} + \mathbf{P} (\mathbf{S}^{-1} \mathcal{L} + \mathbf{C}) = 0. \quad (25)$$

4. WAVES IN TWO DIMENSIONS

In two dimensions the conservative system is

$$\frac{\partial \tilde{\mathbf{U}}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} + \mathbf{C}'_x + \mathbf{C}'_y = 0, \quad (26)$$

with \mathbf{C}'_x and \mathbf{C}'_y representing non-derivative terms, as before. (Only the sum of the \mathbf{C}' terms matters; the sum has been partitioned into two terms to retain consistency with the one dimensional case.) We have the relations

$$\frac{\partial \tilde{\mathbf{U}}}{\partial t} = \mathbf{P} \frac{\partial \mathbf{U}}{\partial t}, \quad (27)$$

$$\frac{\partial \mathbf{F}}{\partial x} = \mathbf{Q} \frac{\partial \mathbf{U}}{\partial x}, \quad \frac{\partial \mathbf{G}}{\partial y} = \mathbf{R} \frac{\partial \mathbf{U}}{\partial y}, \quad (28)$$

$$\mathbf{A} = \mathbf{P}^{-1} \mathbf{Q}, \quad \mathbf{B} = \mathbf{P}^{-1} \mathbf{R}, \quad (29)$$

$$\mathbf{C}_x = \mathbf{P}^{-1} \mathbf{C}'_x, \quad \mathbf{C}_y = \mathbf{P}^{-1} \mathbf{C}'_y, \quad (30)$$

which relate the conservative form of (26) to the primitive form

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x} + \mathbf{B} \frac{\partial \mathbf{U}}{\partial y} + \mathbf{C}_x + \mathbf{C}_y = 0. \quad (31)$$

Now let \mathbf{l}_i , \mathbf{r}_i , and λ_i be the left and right eigenvectors and eigenvalues of \mathbf{A} . Similarly, let \mathbf{m}_i , \mathbf{s}_i , and μ_i be the left and right eigenvectors and eigenvalues of \mathbf{B} . Then the matrices \mathbf{A} and \mathbf{B} can be put in the diagonal forms $\mathbf{\Lambda}$ and \mathbf{M} by the similarity transformations

$$\mathbf{SAS}^{-1} = \mathbf{\Lambda}, \quad \mathbf{TBT}^{-1} = \mathbf{M}. \quad (32)$$

The rows of \mathbf{S} (\mathbf{T}) are the left eigenvectors \mathbf{l}_i (\mathbf{m}_i), the columns of \mathbf{S}^{-1} (\mathbf{T}^{-1}) are the right eigenvectors \mathbf{r}_i (\mathbf{s}_i), and $\mathbf{\Lambda}$ (\mathbf{M}) is the diagonal matrix of eigenvalues λ_i (μ_i). Then Eq. (31) can be rewritten as

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{S}^{-1} \mathbf{\Lambda} \mathbf{S} \frac{\partial \mathbf{U}}{\partial x} + \mathbf{T}^{-1} \mathbf{M} \mathbf{T} \frac{\partial \mathbf{U}}{\partial y} + \mathbf{C}_x + \mathbf{C}_y = 0, \quad (33)$$

which is as close to the characteristic form in one dimension as we can come unless \mathbf{S} and \mathbf{T} are the same (i.e., unless \mathbf{A} and \mathbf{B} are simultaneously diagonalizable), and which will be referred to as a characteristic form due to the presence of the diagonal characteristic velocity matrices.

5. NONREFLECTING BOUNDARY CONDITIONS IN TWO DIMENSIONS

The two dimensional problem allows for an arbitrary number of boundary points, since the boundary is now a curve enclosing a two dimensional space. Let the spatial coordinates be (x, y) , in a general curvilinear coordinate system (not necessarily rectangular). The solution U_{ij} is obtained at the points (x_i, y_j) on a rectangular grid with equal spacings $(\Delta x, \Delta y)$ between successive points in each direction. Interior points have $0 \leq i \leq I, 0 \leq j \leq J$. The boundaries form a rectangle in the xy plane, and each side of the rectangle consists of one or more layers of boundary points. The boundary surfaces intersect at four corners, each of which consists of one or more corner points. Away from the corners, each boundary point has an associated normal and tangential direction (there would be two tangential directions in three dimensions), while at the corner points each direction is normal.

The original conservative system of equations is given in (26). For definiteness, let us consider the y boundaries, defined by the surfaces $y = \text{constant}$, which have the index values $j < 0$ or $j > J$. Then the x derivative, which is in the tangential direction, can be evaluated numerically as an interior term. The y term is in the normal direction, however, and must be put in characteristic form so that the appropriate boundary conditions can be imposed. Thus we write Eq. (26) as

$$\frac{\partial \tilde{\mathbf{U}}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \mathbf{C}'_x + \mathbf{P} \left(\mathbf{T}^{-1} \mathbf{M} \mathbf{T} \frac{\partial \mathbf{U}}{\partial y} + \mathbf{C}_y \right) = 0 \quad (34)$$

at the y boundaries. Abbreviating the quantity in parentheses as $-\partial \mathbf{U} / \partial t_y$, we must evaluate $\partial \mathbf{U} / \partial t_y$, as given by

$$\mathbf{T} \frac{\partial \mathbf{U}}{\partial t_y} + \mathbf{M} \mathbf{T} \frac{\partial \mathbf{U}}{\partial y} + \mathbf{T} \mathbf{C}_y = 0, \quad (35)$$

to provide boundary conditions for (34) at y boundaries. Next define the quantity \mathcal{M}_k :

$$\mathcal{M}_k = \begin{cases} \mu_k \mathbf{m}_k \frac{\partial \mathbf{U}}{\partial y} & \text{for outgoing waves,} \\ 0 & \text{for incoming waves,} \end{cases} \quad (36)$$

and compute $\partial \mathbf{U} / \partial t_y$ from

$$\mathbf{m}_k \frac{\partial \mathbf{U}}{\partial t_y} + \mathcal{M}_k + \mathbf{m}_k \mathbf{C}_y = 0. \quad (37)$$

The spatial derivative in (36) is approximated by the one-sided difference formulas

$$\left. \frac{\partial \mathbf{U}}{\partial y} \right|_{ij} = \frac{1}{\Delta y} (\mathbf{U}_{ij+1} - \mathbf{U}_{ij}), \quad j < 0, \quad (38)$$

$$= \frac{1}{\Delta y} (\mathbf{U}_{ij} - \mathbf{U}_{ij-1}), \quad j > J. \quad (39)$$

Given $\partial \mathbf{U} / \partial t_y$, we compute $\partial \tilde{\mathbf{U}} / \partial t$ from

$$\frac{\partial \tilde{\mathbf{U}}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \mathbf{C}'_x = \mathbf{P} \frac{\partial \mathbf{U}}{\partial t_y}. \quad (40)$$

At the x boundaries the y derivatives are evaluated in conservative form by centered difference approximations, while the x direction terms are put in characteristic form as above. At the corners both directions are normal, and all terms are put in characteristic form, as in (33).

6. NUMERICAL SOLUTION OF THE INTERIOR PROBLEM

The problems considered in this paper are of two types. The first is the one dimensional fluid dynamics problem, in either rectangular or spherical coordinates. The second is a two dimensional problem in rectangular coordinates. In both cases the solutions may be discontinuous, and it is necessary to add dissipative terms to the finite difference approximations in order to damp nonphysical oscillations around the discontinuities. (These oscillations occur because central difference approximations are made to the spatial derivatives in the fluid equations.) The dissipative terms vanish in the limit of zero grid spacing, but diffuse sharp gradients when the spacing is nonzero. The equations, dissipative terms, and numerical methods are more thoroughly discussed in Ref. [8].

The one dimensional system, in conservative form and with dissipative terms included, may be written

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^n} \frac{\partial}{\partial r} (r^n \rho u) = \frac{1}{r^n} \frac{\partial}{\partial r} \left(r^n \varepsilon \Delta r \frac{\partial \rho}{\partial r} \right), \quad (41)$$

$$\frac{\partial m}{\partial t} + \frac{1}{r^n} \frac{\partial}{\partial r} (r^n m u) + \frac{\partial p}{\partial r} = \frac{1}{r^n} \frac{\partial}{\partial r} \left(r^n \varepsilon \Delta r \frac{\partial m}{\partial r} \right) - n \frac{\varepsilon \Delta r}{r^2} m, \quad (42)$$

$$\frac{\partial e}{\partial t} + \frac{1}{r^n} \frac{\partial}{\partial r} [r^n (e + p) u] = \frac{1}{r^n} \frac{\partial}{\partial r} \left(r^n \varepsilon \Delta r \frac{\partial e}{\partial r} \right), \quad (43)$$

where ρ is the density, u the velocity, m the momentum density ($m = \rho u$), e the

energy density ($e = \frac{1}{2} \rho u^2 + p/(\gamma - 1)$), and p the pressure of the fluid. The equation of state can be written

$$p = (\gamma - 1) \left(e - \frac{1}{2} \rho u^2 \right), \quad (44)$$

where γ is the constant ratio of specific heats. The spatial coordinate is r , and the coordinate system is specified by n ($n=0, 1$, or 2 for rectangular, cylindrical, or spherical coordinates, respectively).

The dissipation coefficient $\varepsilon = O(\Delta r)$ in smooth regions and is given by [8]

$$\varepsilon_{i+1/2} = \frac{1}{2} k \frac{\Delta r}{\Delta t} (v_i + v_{i+1}), \quad (45)$$

$$v_i = \frac{|p_{i+1} - 2p_i + p_{i-1}|}{|p_{i+1} - p_i| + |p_i - p_{i-1}|}, \quad (46)$$

$$\varepsilon_i = \frac{1}{2} (\varepsilon_{i-1/2} + \varepsilon_{i+1/2}). \quad (47)$$

The k value determines the maximum amount of dissipation to be added, and is set by the user.

The spatial derivatives on the left side of the equals signs in Eqs. (41)–(43) are evaluated by the fourth order approximation

$$\left. \frac{\partial f}{\partial r} \right|_i = \frac{1}{12\Delta r} [8(f_{i+1} - f_{i-1}) - (f_{i+2} - f_{i-2})] + O(\Delta r^4), \quad (48)$$

while the divergence terms on the right are evaluated by

$$\begin{aligned} & \left. \frac{\partial}{\partial r} \left(r^n \varepsilon \Delta r \frac{\partial f}{\partial r} \right) \right|_i \\ &= \frac{1}{\Delta r} [r_{i+1/2}^n \varepsilon_{i+1/2} (f_{i+1} - f_i) - r_{i-1/2}^n \varepsilon_{i-1/2} (f_i - f_{i-1})] \\ &= O(\Delta r^2). \end{aligned} \quad (49)$$

Although the overall accuracy of the approximations is second order, the fourth order approximation to the spatial derivatives yields sharper jumps at shock waves than do second order approximations.

Replacing the spatial derivatives with the above approximations yields a set of equations of the form

$$\frac{d\tilde{\mathbf{U}}_i}{dt} = (P\tilde{\mathbf{U}})_i, \quad (50)$$

where $\tilde{\mathbf{U}}$ is the vector of conservative unknowns (ρ , m , and e for the fluid equations), and P is a nonlinear operator. (Equation (50) holds at the boundary as well as in the interior, but the P operator is different at the boundary and interior points. The details of the finite differencing at the boundary points are given in Section 8.)

Equation (50) is a set of coupled ordinary differential equations, which may be integrated from time level t^n to level $t^{n+1} = t^n + \Delta t$ by the following 4 step, second order method [9]:

$$\begin{aligned}\tilde{\mathbf{U}}_i^{(1)} &= \tilde{\mathbf{U}}_i^n + \frac{1}{4} \Delta t (P\tilde{\mathbf{U}}^n)_i, \\ \mathbf{U}_i^{(2)} &= \tilde{\mathbf{U}}_i^n + \frac{1}{3} \Delta t (P\tilde{\mathbf{U}}^{(1)})_i, \\ \tilde{\mathbf{U}}_i^{(3)} &= \tilde{\mathbf{U}}_i^n + \frac{1}{2} \Delta t (P\tilde{\mathbf{U}}^{(2)})_i, \\ \tilde{\mathbf{U}}_i^{n+1} &= \tilde{\mathbf{U}}_i^n + \Delta t (P\tilde{\mathbf{U}}^{(3)})_i.\end{aligned}\tag{51}$$

(This method was chosen because it requires a relatively small amount of storage, and is particularly simple to implement in a computer program, as one repeats essentially the same calculation four times to complete each time step.) The boundary equations (25) are integrated along with the interior equations, using the same time stepping scheme.

The two dimensional system, also in conservative form and containing dissipative terms, but in rectangular coordinates, is

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u_x) + \frac{\partial}{\partial y} (\rho u_y) \\ = \frac{\partial}{\partial x} \left(\varepsilon_x \Delta x \frac{\partial \rho}{\partial x} \right) + \frac{\partial}{\partial y} \left(\varepsilon_y \Delta y \frac{\partial \rho}{\partial y} \right),\end{aligned}\tag{52}$$

$$\begin{aligned}\frac{\partial m_x}{\partial t} + \frac{\partial}{\partial x} (m_x u_x) + \frac{\partial}{\partial y} (m_x u_y) + \frac{\partial p}{\partial x} \\ = \frac{\partial}{\partial x} \left(\varepsilon_x \Delta x \frac{\partial m_x}{\partial x} \right) + \frac{\partial}{\partial y} \left(\varepsilon_y \Delta y \frac{\partial m_x}{\partial y} \right),\end{aligned}\tag{53}$$

$$\begin{aligned}\frac{\partial m_y}{\partial t} + \frac{\partial}{\partial x} (m_y u_x) + \frac{\partial}{\partial y} (m_y u_y) + \frac{\partial p}{\partial y} \\ = \frac{\partial}{\partial x} \left(\varepsilon_x \Delta x \frac{\partial m_y}{\partial x} \right) + \frac{\partial}{\partial y} \left(\varepsilon_y \Delta y \frac{\partial m_y}{\partial y} \right),\end{aligned}\tag{54}$$

$$\begin{aligned}\frac{\partial e}{\partial t} + \frac{\partial}{\partial x} [(e+p) u_x] + \frac{\partial}{\partial y} [(e+p) u_y] \\ = \frac{\partial}{\partial x} \left(\varepsilon_x \Delta x \frac{\partial e}{\partial x} \right) + \frac{\partial}{\partial y} \left(\varepsilon_y \Delta y \frac{\partial e}{\partial y} \right),\end{aligned}\tag{55}$$

where (m_x, m_y) and (u_x, u_y) are the momentum density and velocity vectors, respectively. The same finite difference approximations for spatial derivatives as above are made, with ε_x and ε_y depending on the grid spacing and pressure gradients in the x and y directions, as in (45) and (46) [8]. The equation of state is

$$p = (\gamma - 1)[e - \frac{1}{2} \rho (u_x^2 + u_y^2)]. \quad (56)$$

7. CHARACTERISTIC EQUATIONS FOR FLUID DYNAMICS

The boundary conditions require that the fluid equations be put in characteristic form at the boundaries, so we begin the boundary specification for fluid dynamics problems by finding the characteristic form for the fluid equations.

In the one dimensional case we can write the fluid equations in the form of Eq. (2) with

$$\mathbf{U} = \begin{pmatrix} \rho \\ u \\ s \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} u & \rho & 0 \\ \frac{c^2}{\rho} & u & \frac{p}{\rho s} \\ 0 & 0 & u \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \frac{n}{r} \rho u \\ r \\ 0 \\ 0 \end{pmatrix}, \quad (57)$$

where s is a measure of the entropy

$$s = p\rho^{-\gamma}, \quad (58)$$

and c is the speed of sound

$$c^2 = \gamma p / \rho. \quad (59)$$

(The dissipative terms are set to zero at the boundaries.)

The eigenvalues of \mathbf{A} are

$$\lambda_1 = u - c, \quad \lambda_2 = u, \quad \lambda_3 = u + c, \quad (60)$$

and the left eigenvectors are

$$\mathbf{I}_1 = \left(-c, \rho, -\frac{p}{sc} \right), \quad \mathbf{I}_2 = (0, 0, 1), \quad \mathbf{I}_3 = \left(c, \rho, \frac{p}{sc} \right). \quad (61)$$

Taking s as a primitive variable simplifies the eigenvalue calculation, but is inconvenient for numerical work. Therefore we eliminate s in favor of ρ and p and get the characteristic equations

$$\frac{\partial p}{\partial t} - \rho c \frac{\partial u}{\partial t} + \lambda_1 \left(\frac{\partial p}{\partial r} - \rho c \frac{\partial u}{\partial r} \right) + \frac{n}{r} \rho c^2 u = 0, \quad (62)$$

$$\frac{\partial p}{\partial t} - c^2 \frac{\partial \rho}{\partial t} + \lambda_2 \left(\frac{\partial p}{\partial r} - c^2 \frac{\partial \rho}{\partial r} \right) = 0, \quad (63)$$

$$\frac{\partial p}{\partial t} + \rho c \frac{\partial u}{\partial t} + \lambda_3 \left(\frac{\partial p}{\partial r} + \rho c \frac{\partial u}{\partial r} \right) + \frac{n}{r} \rho c^2 u = 0, \quad (64)$$

where the new primitive variable vector \mathbf{U} has components ρ , u , and p .

Given $\partial \mathbf{U} / \partial t$ at the boundaries, $\partial \tilde{\mathbf{U}} / \partial t$ is obtained from (3) by

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial t}, \quad (65)$$

$$\frac{\partial m}{\partial t} = u \frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial t}, \quad (66)$$

$$\frac{\partial e}{\partial t} = \frac{1}{2} u^2 \frac{\partial \rho}{\partial t} + \rho u \frac{\partial u}{\partial t} + \frac{1}{\gamma - 1} \frac{\partial p}{\partial t}. \quad (67)$$

In two dimensions the fluid equations may be written in the form of (33) with

$$\mathbf{U} = \begin{pmatrix} \rho \\ u_x \\ u_y \\ s \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} u_x & \rho & 0 & 0 \\ \frac{c^2}{\rho} & u_x & 0 & \frac{p}{\rho s} \\ 0 & 0 & u_x & 0 \\ 0 & 0 & 0 & u_x \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} u_y & 0 & \rho & 0 \\ 0 & u_y & 0 & 0 \\ \frac{c^2}{\rho} & 0 & u_y & \frac{p}{\rho s} \\ 0 & 0 & 0 & u_y \end{pmatrix}, \quad \mathbf{C}_x = \mathbf{C}_y = 0. \quad (68)$$

The eigenvalues of \mathbf{A} and \mathbf{B} are

$$\lambda_1 = u_x - c, \quad \lambda_2 = \lambda_3 = u_x, \quad \lambda_4 = u_x + c, \quad (69)$$

$$\mu_1 = u_y - c, \quad \mu_2 = \mu_3 = u_y, \quad \mu_4 = u_y + c. \quad (70)$$

The y direction characteristic terms, in the form of (35), are

$$\frac{\partial p}{\partial t_y} - \rho c \frac{\partial u_y}{\partial t_y} + \mu_1 \left(\frac{\partial p}{\partial y} - \rho c \frac{\partial u_y}{\partial y} \right) = 0, \quad (71)$$

$$\frac{\partial u_x}{\partial t_y} + \mu_2 \frac{\partial u_x}{\partial y} = 0, \quad (72)$$

$$\frac{\partial p}{\partial t_y} - c^2 \frac{\partial \rho}{\partial t_y} + \mu_3 \left(\frac{\partial p}{\partial y} - c^2 \frac{\partial \rho}{\partial y} \right) = 0, \quad (73)$$

$$\frac{\partial p}{\partial t_y} + \rho c \frac{\partial u_y}{\partial t_y} + \mu_4 \left(\frac{\partial p}{\partial y} + \rho c \frac{\partial u_y}{\partial y} \right) = 0. \quad (74)$$

An analogous set holds for the x direction.

The conservative and primitive time derivatives are related by

$$\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial t}, \quad (75)$$

$$\frac{\partial m_x}{\partial t} = u_x \frac{\partial \rho}{\partial t} + \rho \frac{\partial u_x}{\partial t}, \quad (76)$$

$$\frac{\partial m_y}{\partial t} = u_y \frac{\partial \rho}{\partial t} + \rho \frac{\partial u_y}{\partial t}, \quad (77)$$

$$\frac{\partial e}{\partial t} = \frac{1}{2} (u_x^2 + u_y^2) \frac{\partial \rho}{\partial t} + \rho \left(u_x \frac{\partial u_x}{\partial t} + u_y \frac{\partial u_y}{\partial t} \right) + \frac{1}{\gamma - 1} \frac{\partial p}{\partial t}. \quad (78)$$

In the one dimensional case, we solve Eqs. (41)–(43) in the interior using centered finite difference approximations for the spatial derivatives, and an explicit ordinary differential equation solver to integrate the time derivatives of the conservative variables. The interior algorithm requires data at the boundary points, which are obtained by solving the combined characteristic and nonreflecting equations at those points. The details of the boundary calculations are given below.

We first write the boundary equations as

$$\frac{dp_i}{dt} - \rho_i c_i \frac{du_i}{dt} + \mathcal{L}_{1i} + \frac{n}{r_i} \rho_i c_i^2 u_i = 0, \quad (79)$$

$$\frac{dp_i}{dt} - c_i^2 \frac{dp_i}{dt} + \mathcal{L}_{2i} = 0, \quad (80)$$

$$\frac{dp_i}{dt} + \rho_i c_i \frac{du_i}{dt} + \mathcal{L}_{3i} + \frac{n}{r_i} \rho_i c_i^2 u_i = 0, \quad (81)$$

where each \mathcal{L}_{ki} is set to zero if λ_k at r_i is directed inward (nonreflecting condition), or is computed according to the characteristic equations if λ_k at r_i is directed outward:

$$\mathcal{L}_{1i} = (u_i - c_i) \frac{1}{\Delta r} [p_{i+1} - p_i - \rho_i c_i (u_{i+1} - u_i)], \quad i < 0, \quad u_i - c_i < 0, \quad (82)$$

$$= (u_i - c_i) \frac{1}{\Delta r} [p_i - p_{i-1} - \rho_i c_i (u_i - u_{i-1})], \quad i > I, \quad u_i - c_i > 0; \quad (83)$$

$$\mathcal{L}_{2i} = u_i \frac{1}{\Delta r} [p_{i+1} - p_i - c_i^2 (\rho_{i+1} - \rho_i)], \quad i < 0, \quad u_i < 0, \quad (84)$$

$$= u_i \frac{1}{\Delta r} [p_i - p_{i-1} - c_i^2 (\rho_i - \rho_{i-1})], \quad i > I, \quad u_i > 0; \quad (85)$$

$$\mathcal{L}_{3i} = (u_i + c_i) \frac{1}{\Delta r} [p_{i+1} - p_i + \rho_i c_i (u_{i+1} - u_i)], \quad i < 0, \quad u_i + c_i < 0, \quad (86)$$

$$= (u_i + c_i) \frac{1}{\Delta r} [p_i - p_{i-1} + \rho_i c_i (u_i - u_{i-1})], \quad i > I, \quad u_i + c_i > 0. \quad (87)$$

Given the \mathcal{L} values, the time derivatives of the the primitive variables are

$$\frac{dp_i}{dt} = -\frac{1}{2} (\mathcal{L}_{3i} + \mathcal{L}_{1i}) - \frac{n}{r_i} \rho_i c_i^2 u_i, \quad (88)$$

$$\frac{du_i}{dt} = -\frac{1}{2\rho_i c_i} (\mathcal{L}_{3i} - \mathcal{L}_{1i}), \quad (89)$$

$$\frac{d\rho_i}{dt} = \frac{1}{c_i^2} \left(\frac{dp_i}{dt} + \mathcal{L}_{2i} \right), \quad (90)$$

and Eqs. (66) and (67) provide $dm_{i\parallel}/dt$ and $de_{i\parallel}/dt$ at the boundaries, to be integrated in time along with the interior values.

The two dimensional case is similar. We solve Eqs. (52)–(55) in the interior, using centered finite difference approximations for the spatial derivatives, and an explicit integration method for the time derivatives. We now have four boundaries, defined by $x = x_{\min}$, $x = x_{\max}$, $y = y_{\min}$, and $y = y_{\max}$. Since all four boundaries are treated in a similar fashion, it will be sufficient to look at the $y = \text{constant}$ boundaries.

We begin by writing the fluid equations at the y boundary points as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u_x) - \frac{\partial \rho}{\partial t_y} = \frac{\partial}{\partial x} \left(\varepsilon_x \Delta x \frac{\partial \rho}{\partial x} \right), \quad (91)$$

$$\frac{\partial m_x}{\partial t} + \frac{\partial}{\partial x} (m_x u_x) + \frac{\partial p}{\partial x} - \frac{\partial m_x}{\partial t_y} = \frac{\partial}{\partial x} \left(\varepsilon_x \Delta x \frac{\partial m_x}{\partial x} \right), \quad (92)$$

$$\frac{\partial m_y}{\partial t} + \frac{\partial}{\partial x} (m_y u_x) - \frac{\partial m_y}{\partial t_y} = \frac{\partial}{\partial x} \left(\varepsilon_x \Delta x \frac{\partial m_y}{\partial x} \right), \quad (93)$$

$$\frac{\partial e}{\partial t} + \frac{\partial}{\partial x} [(e + p) u_x] - \frac{\partial e}{\partial t_y} = \frac{\partial}{\partial x} \left(\varepsilon_x \Delta x \frac{\partial e}{\partial x} \right), \quad (94)$$

where the $\partial \tilde{\mathbf{U}}/\partial t_y$ terms are the contributions to $\partial \tilde{\mathbf{U}}/\partial t$ due to derivatives in the normal (y) direction. The x derivatives are in the tangential direction, relative to the boundary, and are evaluated just as in the interior.

We need to compute the $\partial \tilde{\mathbf{U}}/\partial t_y$ terms in (91)–(94), and begin by writing Eqs. (71)–(74) in finite difference form, as

$$\frac{dp_{ij}}{dt_y} - \rho_{ij}c_{ij} \frac{du_{yij}}{dt_y} + \mathcal{M}_{1ij} = 0, \quad (95)$$

$$\frac{du_{xij}}{dt_y} + \mathcal{M}_{2ij} = 0, \quad (96)$$

$$\frac{dp_{ij}}{dt_y} - c_{ij}^2 \frac{d\rho_{ij}}{dt_y} + \mathcal{M}_{3ij} = 0, \quad (97)$$

$$\frac{dp_{ij}}{dt_y} + \rho_{ij}c_{ij} \frac{du_{yij}}{dt_y} + \mathcal{M}_{4ij} = 0, \quad (98)$$

where each \mathcal{M}_{kij} is set to zero if the local y direction characteristic velocity, μ_k , is directed inward (nonreflecting condition), or is computed according to the characteristic forms below if μ_k is directed outward:

$$\begin{aligned} \mathcal{M}_{1ij} &= (u_{yij} - c_{ij}) \frac{1}{\Delta y} [p_{ij+1} - p_{ij} - \rho_{ij}c_{ij}(u_{yij+1} - u_{yij})], \\ & \quad j < 0, \quad u_{yij} - c_{ij} < 0, \end{aligned} \quad (99)$$

$$\begin{aligned} &= (u_{yij} - c_{ij}) \frac{1}{\Delta y} [p_{ij} - p_{ij-1} - \rho_{ij}c_{ij}(u_{yij} - u_{yij-1})], \\ & \quad j > J, \quad u_{yij} - c_{ij} > 0; \end{aligned} \quad (100)$$

$$\mathcal{M}_{2ij} = u_{yij} \frac{1}{\Delta y} (u_{xij+1} - u_{xij}), \quad j < 0, \quad u_{yij} < 0, \quad (101)$$

$$= u_{yij} \frac{1}{\Delta y} (u_{xij} - u_{xij-1}), \quad j > J, \quad u_{yij} > 0; \quad (102)$$

$$\mathcal{M}_{3ij} = u_{yij} \frac{1}{\Delta y} [p_{ij+1} - p_{ij} - c_{ij}^2(\rho_{ij+1} - \rho_{ij})], \quad j < 0, \quad u_{yij} < 0, \quad (103)$$

$$= u_{yij} \frac{1}{\Delta y} [p_{ij} - p_{ij-1} - c_{ij}^2(\rho_{ij} - \rho_{ij-1})], \quad j > J, \quad u_{yij} > 0; \quad (104)$$

$$\begin{aligned} \mathcal{M}_{4ij} &= (u_{yij} + c_{ij}) \frac{1}{\Delta y} [p_{ij+1} - p_{ij} + \rho_{ij}c_{ij}(u_{yij+1} - u_{yij})], \\ & \quad j < 0, \quad u_{yij} + c_{ij} < 0, \end{aligned} \quad (105)$$

$$\begin{aligned} &= (u_{yij} + c_{ij}) \frac{1}{\Delta y} [p_{ij} - p_{ij-1} + \rho_{ij}c_{ij}(u_{yij} - u_{yij-1})], \\ & \quad j > J, \quad u_{yij} + c_{ij} > 0. \end{aligned} \quad (106)$$

Given the \mathcal{M} values, we compute $\partial U/\partial t_y$ from

$$\frac{dp_{ij}}{dt_y} = -\frac{1}{2}(\mathcal{M}_{4ij} + \mathcal{M}_{1ij}), \quad (107)$$

$$\frac{du_{yij}}{dt_y} = -\frac{1}{2\rho_{ij}c_{ij}}(\mathcal{M}_{4ij} - \mathcal{M}_{1ij}), \quad (108)$$

$$\frac{du_{xij}}{dt_y} = -\mathcal{M}_{2ij}, \quad (109)$$

$$\frac{d\rho_{ij}}{dt_y} = \frac{1}{c_{ij}^2} \left(\frac{dp_{ij}}{dt_y} + \mathcal{M}_{3ij} \right). \quad (110)$$

Finally, dm_{xij}/dt_y , dm_{yij}/dt_y , and de_{ij}/dt_y are calculated using (76)–(78), and their values are used in the finite difference approximations to Eqs. (91)–(94). A similar process is followed at the x boundaries.

9. TEST PROBLEMS

The one dimensional problems are evaluated on a grid of unit length, divided into 100 subintervals ($I=100$, $\Delta r=0.01$). The value of γ in the equation of state is $5/3$ throughout. The figures show the analytical solutions (solid lines) and numerical solutions (dots) for the density ρ , the pressure p , the momentum density m , and the velocity u , at a time t . The Courant number λ ,

$$\begin{aligned} \lambda &= \frac{c_r \Delta t}{\Delta r} && (1 \text{ dimension}), \\ &= \left(\frac{c_x}{\Delta x} + \frac{c_y}{\Delta y} \right) \Delta t && (2 \text{ dimensions}), \end{aligned} \quad (111)$$

is set to 1 throughout, where $c_x = \max(|u_x| + c)$ over the grid (and similarly for c_y and c_r).

The first problem is a single shock wave moving into a uniform stationary medium. The shock starts at $x=0.5$ and has a positive velocity. The pre-shock state has $\rho=p=1$ and $u=0$. The problem is determined by the pre- and post-shock values of ρ , p , and u , and the shock velocity V_s , and must satisfy the three shock jump conditions [10]. Thus we have one more free parameter, chosen to be the Mach number $M=u/c$ of the post-shock flow. The Mach number is related to the ratio $R=p_s/p_u$ of the shocked to unshocked pressures by

$$M^2 = \frac{2}{\gamma R} \frac{(R-1)^2}{\gamma + 1 + (\gamma-1)R}. \quad (112)$$

As $R \rightarrow \infty$, $M^2 \rightarrow M_{\max}^2 = 2/[\gamma(\gamma-1)]$, where $M_{\max} = 1.3416$ for $\gamma = 5/3$.

TABLE I
Reflections for Subsonic Shocks

Mach number M	Pressure ratio $R = p_s/p_u$	Relative error (%) $100 \times (p_{\text{num}} - p_{\text{an}})/p_{\text{an}}$
0.50	2.504	0.08
0.60	3.096	0.19
0.70	3.891	0.33
0.80	5.000	0.48
0.90	6.635	0.68
0.92	7.058	0.71
0.94	7.524	0.74
0.95	7.775	0.77
0.96	8.040	0.78
0.97	8.319	0.81
0.98	8.614	0.82
0.99	8.926	0.82
0.995	9.089	0.80
1.00	9.257	0.77

A reflection is generated when an outgoing subsonic shock wave crosses a boundary, as demonstrated by Hedstrom [7]. The reflection takes the form of a constant amplitude perturbation to the post-shock solution, which travels inward from the boundary at the speed of sound relative to the moving fluid. A convenient measure of the reflection is the relative error in the pressure after the shock has crossed the boundary, defined as the difference between the numerical and analytical pressure values, divided by the analytical pressure. Table I gives the pressure ratio R and reflections as a function of the Mach number, for subsonic shock waves modeled with $k = 0.35$. (A significantly smaller value of k results in large oscillations near the shock, while a larger value spreads the shock jump out over many grid points.)

The reflections are small, nowhere exceeding 1%. The worst case is the $M = 0.98$ shock, with a reflection of 0.82%. It is interesting to note that the reflection decreases as M increases from 0.99 to 1.0, although the shock jump increases.

Hedstrom [7] observed a reflection of 12% in the velocity profile of his Fig. 6. It is not certain why there is such a large discrepancy between his results and those presented here, but a likely culprit is the mismatch between his interior and boundary methods. He used the Lax-Wendroff method for the interior. At the boundary, he used first order approximations to the spatial derivatives, and

$$\left. \frac{\partial \mathbf{U}}{\partial t} \right|_i^n = \frac{1}{\Delta t} (\mathbf{U}_i^{n+1} - \mathbf{U}_i^n) + O(\Delta t) \quad (113)$$

for the time derivatives. The two time integration methods are quite different. In

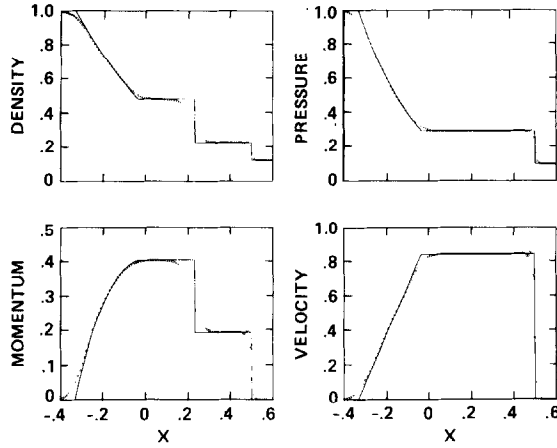


FIG. 1. Shock tube at step 60.

contrast, the four step time integration scheme of (51) was used for both interior and boundary points here.

When the flow behind the shock is supersonic ($M > 1$) all characteristics point to the right, and no signals can propagate to the left. Thus no reflections can be produced, and none are observed.

The next example is the shock tube problem, frequently used as a test for hydrodynamical codes (as in Sod [11]), and whose solution is given by Thompson [8]. At time $t=0$ the system consists of two spatially constant, stationary states, adjoining at $x=0$. The left state ($x < 0$) has $\rho = p = 1$, while the right state has $\rho = 0.125$, $p = 0.1$. As time progresses, a rarefaction wave forms and moves to the left, while the contact discontinuity and a shock wave move to the right.

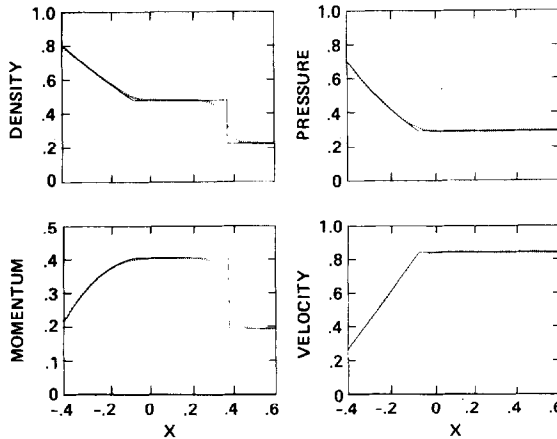


FIG. 2. Shock tube at step 100.

Figure 1 shows the solution at time step 60, $t = 0.265$, with $k = 0.3$, just before any waves reach the boundaries. Figure 2 shows the same problem at step 100 and $t = 0.438$. The rarefaction wave has passed through the left boundary, and the shock wave has passed through the right boundary. No reflections are visible on either side. The boundaries are well behaved.

The next problem considered is the Sedov solution for a spherically symmetric explosion, described by Landau and Lifshitz [10]. (N.B. Equation (99.10) of the reference should have $v_5 = 2/(\gamma - 2)$.) An amount of energy E is deposited at the origin at time $t = 0$. The resultant explosion produces a self-similar solution bounded by an outgoing spherical shock wave. The density and pressure curves are sharply peaked at the shock, falling off rapidly for decreasing r . The density goes to zero at the origin, while the pressure flattens out and becomes constant with r away from the shock. The velocity is linear near the origin, but steepens somewhat near the shock. The similarity solution is valid as long as the shock is very strong, so that the density jump across the shock achieves its maximum value.

The numerical solution is produced by distributing an amount of energy $E = 1$ over the innermost five grid intervals, in the form of thermal energy. Let the sum of the volume elements for the first five points be

$$vol = 4\pi Ar \sum_{i=0}^5 r_i^2, \tag{114}$$

then the initial pressure is

$$p_i = (\gamma - 1) E/vol, \tag{115}$$

for $i = 0, \dots, 5$. For $i > 5$, $p_i = 10^{-6}$. We also have $\rho = 1$ and $u = 0$ everywhere. The resultant numerical solution is shown in Fig. 3 at step 422, $t = 0.602$, and with $k = 0.14$.

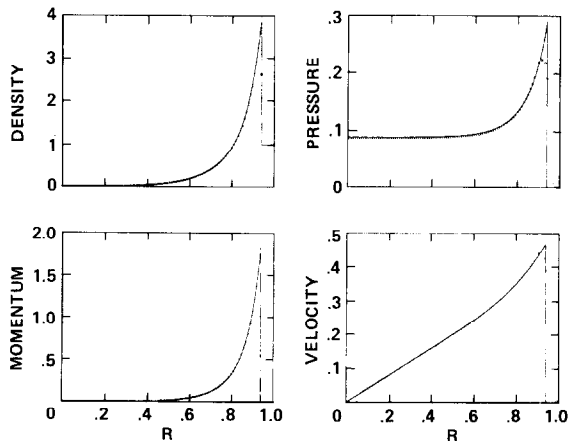


FIG. 3. Spherical explosion at step 422.

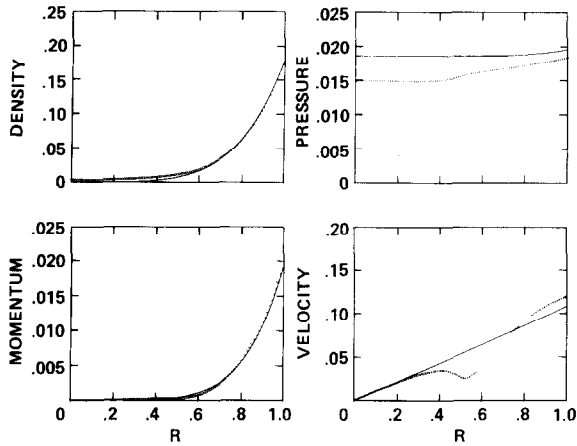


FIG. 4. Spherical explosion at step 1000, magnified.

Figure 4 shows the explosion at step 1000, $t = 2.243$, with the vertical scales magnified. The shock wave has passed out of the domain, and the velocity at the boundary has decreased from supersonic to subsonic. A perturbation has developed at the boundary and is propagating inward, as can be seen most clearly from the velocity curve. At later times the perturbation is reflected from the center and propagates out of the problem, leaving an altered interior state. The pressure is then low everywhere by about 15%, and the velocity is low by about 40%. The discrepancy is presumably due to the fact that the similarity solution for the explosion really does contain an inward propagating wave, which is suppressed by the boundary conditions. The resulting numerical solution is a valid one, but it is not the solution to the explosion problem at late times.

The impossibility of properly specifying boundary conditions for all problems with the nonreflecting prescription is further illustrated by the following problem, the homologous expansion of a uniform medium. At time $t = 0$ the density and pressure are uniform, with $\rho = p = 1$. The velocity is linear, with $u = x/t_0$, t_0 chosen to be 1. The region studied is $-0.5 \leq x \leq 0.5$. The density and pressure decrease with time but remain uniform, while the velocity also decreases but remains linear in x . The flow at the boundaries is always subsonic and directed outward for this set of initial conditions.

The problem has a simple analytical solution, given by

$$\rho = \rho_0 \left(1 + \frac{t}{t_0}\right)^{-1}, \quad (116)$$

$$p = p_0 \left(1 + \frac{t}{t_0}\right)^{-\gamma}, \quad (117)$$

$$u = \frac{x}{t + t_0}. \quad (118)$$

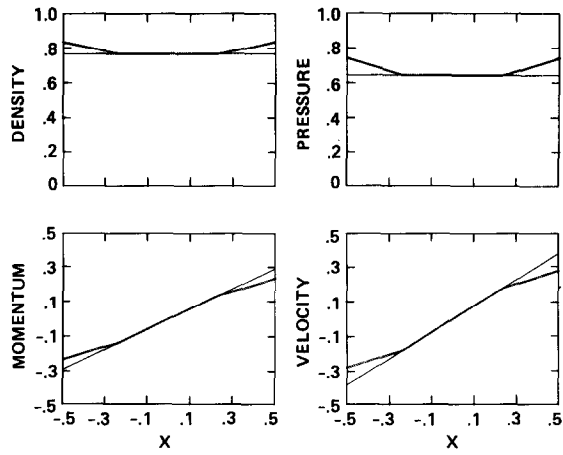


FIG. 5. Homologous expansion at step 50.

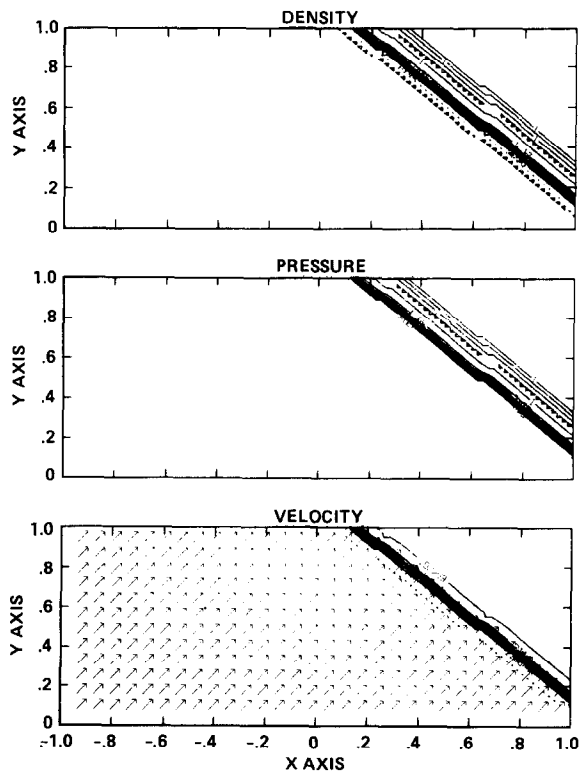


FIG. 6. Angled shock, step 1.

One can verify by direct substitution that this solution does not satisfy the non-reflecting boundary conditions. For example, at the right boundary ($x = b$) the flow is subsonic and directed outward. The two characteristic equations representing outgoing waves (Eqs. (63) and (64)) hold as written, while the absence of an incoming wave is imposed by the nonreflecting condition of Eq. (79) with $\mathcal{L}_1 = 0$, which can be written

$$\frac{\partial p}{\partial t} - \rho c \frac{\partial u}{\partial t} = 0.$$

The analytic solution does not satisfy the nonreflecting condition, so the non-reflecting condition will not produce the desired numerical solution.

Figure 5 shows the expansion problem at step 50, $t = 0.305$, with $k = 0$. The numerical and analytic solutions diverge markedly near the boundaries, although they match well in the middle portion. The discrepancy grows with time until the two solutions disagree everywhere.

We can use information about this particular problem to specify better boundary

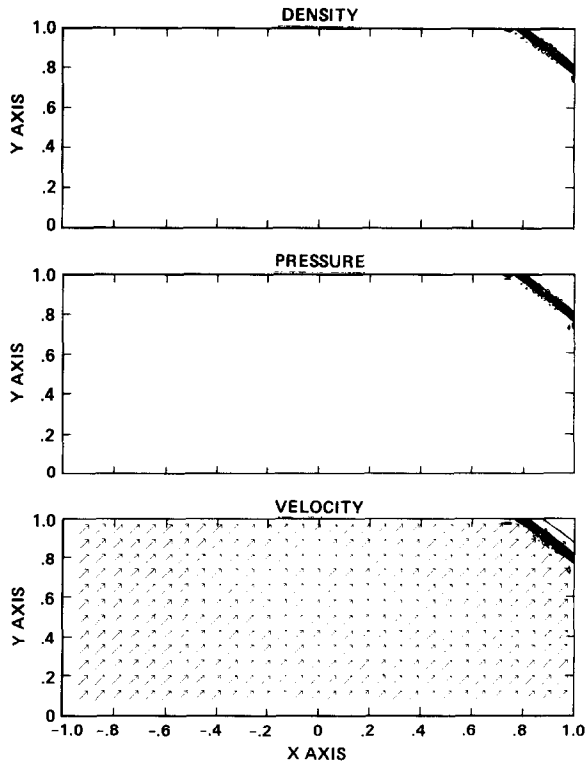


FIG. 7. Angled shock, step 50.

conditions. In particular, the pressure gradient is zero everywhere, and the velocity satisfies

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0, \tag{119}$$

which is in characteristic form and describes outflow at the boundary. If (119) is used in place of the nonreflecting boundary condition, we have three characteristic equations describing outgoing waves. (By throwing out the pressure gradient term, we have excluded sound waves from the problem. The only characteristic velocity left is the fluid velocity. The evolution of the velocity profile then determines the rest of the solution). The resulting numerical solution matches the analytical solution everywhere.

This result is not meant as an endorsement of Eq. (119) as a boundary condition in general (it does nothing useful for the explosion problem), but simply illustrates that nonreflecting boundary conditions cannot be expected to give the desired results on problems which do contain incoming waves. Information specific to such problems may be used to produce more useful boundary conditions.

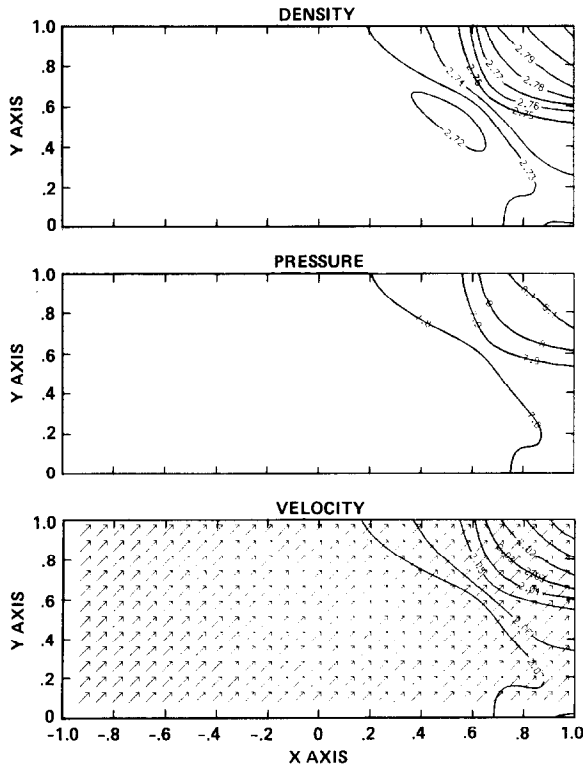


FIG. 8. Angled shock, step 200.

The final test problem is two dimensional. It is a planar Newtonian shock, traveling in rectangular geometry. The grid has $\Delta x = \Delta y = 0.02$. The shock is traveling toward the upper right, at a 45° angle with respect to the x axis. The initial distance R_0 between the shock front and the origin is 0.8. The unshocked density and pressure are $\rho = 1$, $p = 1$, with $u_x = u_y = 0$. The Mach number of the flow behind the shock is $M = 0.95$, picked to roughly maximize reflection errors. The dissipation used is $k = 0.1$.

Contour plots of the density, pressure, and velocity fields in Figs. 6 through 8 show the time evolution of the solution. The figures are at step 1, $t = 0.03$; step 50, $t = 0.14$; and step 200, $t = 0.58$, respectively. Step 1 shows essentially the initial conditions, with a small amount of transient jitter induced by the discontinuous data, which disappears as the solution evolves. Step 50 shows the shock shortly before it reaches the corner. A slight hint of boundary perturbations can be seen near the edges of the shock. Step 200 shows the solution well after the shock has left the grid. The reflection from the corner has propagated inward, and at its peak amounts to about 4% of the post-shock pressure profile. The reflection is greater than in the one dimensional case, perhaps because the corner is subject to reflections from two coordinate directions, but not enough to obscure significant features of the post-shock flow (if there were any).

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